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Markov Systems of Vector-Valued Functions and Disconjugacy

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Let $\mathscr{X}$ be an n-dimensional vector space of functions x from $\mathbb{R}$ to $\mathbb{R}^k$. Suppose I is a real interval and $r_1, ..., r_k$ are nonnegative integers with $r_1 + \cdots + r_k = n$. We investigate conditions on $\mathscr{X}$, I, and $(r_1, ..., r_k)$ which imply that $x = 0$ is the only element of $\mathscr{X}$ whose ith component x_i has r_i zeros in I, $i = 1, ..., k$. When $k = 1$ and the elements of $\mathscr{X}$ are sufficiently smooth, Pólya [*Trans. Amer. Math. Soc.* **29** (1922), 312–324] showed that a sufficient condition is that $\mathscr{X}$ has a basis which is a Markov system on I. Moreover this condition is necessary if the elements of $\mathscr{X}$ are smooth and I is open or closed. This result is generalized to $k \geqslant 1$. Examples show how this approach provides criteria for incompatibility of certain classes of linear homogeneous boundary value problems for ordinary differential equations.

1. INTRODUCTION

Suppose L is a real linear differential operator of order n,

$$L = \sum_{i=0}^{n} a_{n-i} D^i, \tag{1.1}$$

where the coefficients a_i are continuous real-valued functions on a real interval J, $a_0 = 1$, and $D^i u = u^{(i)}$ is the derivative of order i of a function u on J. An operator of this form is *disconjugate* on an interval $I \subset J$ if the only solution u of the differential equation $Lu = 0$ which has n or more zeros in I is the zero solution. For example, the operator D^n is disconjugate on every real interval, while $D^2 + 1$ is disconjugate only on intervals of

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length less than π and on open or half-open intervals of length π. Disconjugate operators have a rich structure which has been the subject of considerable study. An excellent exposition is contained in the book of Coppel [1] where a full set of references will be found.

An operator L of the form (1.1) is *disfocal* on an interval I if

$$Lu=0, \qquad u^{(i-1)}(t_i)=0, \qquad t_i \in I, \quad i=1, ..., n, \tag{1.2}$$

implies $u=0$. It is readily seen from Rolle's theorem that disfocality of L on an interval I implies disconjugacy of L on I; the converse is not true as can be seen by observing that D^2+1 is disfocal only on intervals of length less than $\pi/2$ and on open or half-open intervals of length $\pi/2$. Disfocality has also been the subject of considerable study especially for operators of the form D^n+a_n and generalizations of these, notably by Nehari [17], Elias (cf. references in [2]), Jones [9], and Kim [11].

The operator L is *right-disfocal* on I if, in (1.2), one restricts the sequences of points $t_i \in I$ where successive derivatives of u are zero to be nondecreasing: $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$. Similarly, L is left-disfocal on I if only t_i such that $t_1 \geqslant t_2 \geqslant \cdots \geqslant t_n$ are considered in (1.2). Other orderings of the points t_i might also be considered.

A basic result in the theory of disconjugate operators is due to Pólya [20] (see also [10, pp. 376–378]) and states that L is disconjugate on an interval I if there exist solutions $u_1, ..., u_n$ of $Lu=0$ which form a Markov system; specifically

$$W(u_1, ..., u_p) > 0, \qquad p=1, ..., n, \tag{1.3}$$

on I, where $W(u_1, ..., u_p)=\det[u_i^{(j-1)}]$, $i, j=1, ..., p$, is the Wronskian determinant. This is also a necessary condition for disconjugacy if the interval I is either open or closed. A similar result proved in [15] states that L is right-disfocal on I if there exist solutions $u_1, ..., u_n$ of $Lu=0$ such that

$$W(u_1^{(j-1)}, ..., u_p^{(j-1)}) > 0, \qquad j=1, ..., n-p+1, \quad p=1, ..., n, \tag{1.4}$$

and this condition is necessary for right-disfocality if the interval I is closed. Results similar to (1.3) and (1.4) are obtained in [16] for concepts related to disconjugacy and disfocality.

We observe that in (1.3) the signs of the Wronskians are not particularly important; the nonvanishing of these determinants is the crucial requirement and their signs can always be arranged by replacing some solutions u_i by $-u_i$. In contrast, it is not sufficient that all the Wronskians in (1.4) be nonzero; the pattern of signs exhibited by the Wronskians is also important and is relevant to the order which may be placed on the points t_i in (1.2).

A result of Hartman [6, Corollary 3.1, p. 51] gives a necessary and sufficient condition similar to (1.3) for the factorization of a generalized linear differential operator into a product of first order operators. This result would permit the extension of other results on disconjugacy to such operators. Such an extension is carried out by Nehari in [17] although not in the generality permitted by Hartman's result. Nehari's paper is mainly concerned with questions of disfocality in a general setting, especially for generalized forms of $D^n + a_n$.

Hartman [7] develops the theory of disconjugacy for linear difference operators and derives analogues of many of the fundamental results. These studies have been continued by others; see, for example, Peterson [19].

Questions about nonlinear boundary value problems closely related to the theory of disconjugacy have also had considerable study; see Henderson and Jackson [8].

This paper also has contact with the theory of nonoscillation for equations of the form

$$x' - A(t)x = 0, \tag{1.5}$$

where x is an n-dimensional column vector and $A(t)$ is an $n \times n$ matrix. Such an equation is nonoscillatory on an interval I if each nontrivial solution of (1.5) has at least one component with no zeros in I. This theory was developed by Schwarz [21], London and Schwarz [13], Nehari [18], Kim [12], Friedland [3], and others. A comprehensive bibliography of these authors may be found in [3].

An operator L of the form (1.1) is disconjugate on an interval I if and only if its adjoint is also disconjugate on I (cf. [1, p. 104]). This is a very useful observation in deriving concrete results such as comparison and other criteria for disconjugacy (cf. [14, 16]). To obtain similar results for disfocality, one is led to consider adjoint boundary value problems to questions such as (1.2) in the cases that all the points t_i occur at just two points. The adjoint boundary conditions are then of the form $l_i u(t_i) = 0$ where l_i are not necessarily derivatives but generalized differential operators.

The present study was initiated as an investigation of criteria like (1.3), (1.4) for problems such as (1.2) with the boundary conditions $u^{(i-1)}(t_i) = 0$ replaced by more general differential boundary conditions $l_i u(t_i) = 0$. The techniques developed were found to be applicable to general linear (not necessarily differential) boundary conditions and indeed the idea that the underlying space of objects u was the null set of a linear differential operator L of the form (1.1) decreased in significance. A slightly more abstract formulation of the concepts in fact leads to simplification of the proofs and a wider variety of differential operators being discussed.

2. Notation and Preliminary Definitions

Function spaces. Let $J = (a, b)$, $-\infty \leqslant a < b \leqslant \infty$, and let $\mathscr{X}$ denote a subspace of $C(J \to \mathbb{R}^k)$. The elements of $\mathscr{X}$ will be represented by column vectors. If $x = (x_1, ..., x_k)^T \in \mathscr{X}$, it will be assumed that the components x_i are sufficiently smooth so that zeros of x_i of a specific multiplicity may be considered and, if $t \in J$, $D^j x_i(t)$ denotes the value at t of the jth derivative of x_i.

Partitions. Let $\tau = (\tau_1; ...; \tau_k)$, where each τ_i is a finite (possibly empty) nondecreasing sequence of points in J. The number of entries, counting multiplicities, in τ_i is denoted $|\tau_i|$ and the number of entries in I is $|\tau| = |\tau_1| + \cdots + |\tau_k|$. Then τ is called a partition of $|\tau|$ points in J. Also $|\tau_i(t)|$ denotes the multiplicity of an entry t in τ_i and $|\tau(t)| = |\tau_1(t)| + \cdots + |\tau_k(t)|$ is the number of times t occurs in τ.

Values and zeros. Let $x \in \mathscr{X}$ and let τ be a partition of points in J. The value of x_i at τ, if $|\tau_i| = q \neq 0$, is the column vector $x_i[\tau] \in \mathbb{R}^q$ defined as

$$x_i[\tau] = (x_i(s_1), ..., x_i(s_q))^T, \qquad \text{if} \quad \tau_i = (s_1, ..., s_q)$$

and s_j are all distinct. If an element s of τ_i has multiplicity $|\tau_i(s)| = p$, then the corresponding entries in $x_i[\tau]$ are $x_i(s)$, $Dx_i(s)$, ..., $D^{p-1}x_i(s)$. The value of x at τ is a column vector $x[\tau] \in \mathbb{R}^{|\tau|}$ defined in block form by

$$x[\tau] = (x_1[\tau], ..., x_k[\tau])^T,$$

where only those x_i for which $|\tau_i| \neq 0$ are included. For example, if $k = 2$, then

$$x[t_1, t_2; t_3] = (x_1(t_1), x_1(t_2), x_2(t_3))^T, \qquad \text{if} \quad t_1 < t_2,$$

$$x[t_1, t_2; t_3] = (x_1(t_1), Dx_1(t_1), x_2(t_3))^T, \qquad \text{if} \quad t_1 = t_2.$$

If $|\tau_i| = |\tau_i(t)| = 1$, $i = 1, ..., k$, then $x[\tau] = x[t; ...; t] = (x_1(t), ..., x_k(t))^T = x(t)$ and, if $|\tau| = |\tau_j| = |\tau_j(t)| = p$, for some j, then $x[\tau] = x_j[\tau] = (x_j(t), Dx_j(t), ..., D^{p-1}x_j(t))$. The value $x[\tau]$ is continuous with respect to τ for which $|\tau_i|$ and the multiplicities of the various entries in τ are fixed. An element $x \in \mathscr{X}$ is zero at τ if $x[\tau] = 0$, the zero of $\mathbb{R}^{|\tau|}$. Thus "x is zero at τ" means that $D^{j-1}x_i(t) = 0$, $j = 1, ..., |\tau_i(t)|$, $i = 1, ..., k$, $t \in \tau$.

Generalized Wronskians. Let $X = [x^1, ..., x^n]$ be an n-tuple of elements x^j of $\mathscr{X}$. Let τ be a partition of $q = |\tau|$ points in J. Then $X[\tau]$ is the $q \times n$ matrix whose jth column is $x^j[\tau]$, the value of x^j at τ. Thus, if an element t

of τ_i has multiplicity $|\tau_i(t)| = r_i$, then the corresponding block of r_i rows in $X[\tau]$ is

$$\begin{bmatrix} x_i^1(t), & ..., x_i^n(t) \\ Dx_i^1(t), & ..., Dx_i^n(t) \\ \cdots & \cdots\cdots \\ D^{r_i-1}x_i^1(t), & ..., D^{r_i-1}x_i^n(t) \end{bmatrix}. \tag{2.1}$$

If $t \in J$ and $r = (r_1, ..., r_k)$, where $r_i \geqslant 0$ are integers, such that $|r| = r_1 + \cdots + r_k \leqslant n$, the generalized Wronskian $W(X : r)(t)$ is defined as follows. Consider the partition τ in which the only entry is t with $|\tau_i(t)| = |\tau_i| = r_i$, $i = 1, ..., k$, so that $X[\tau]$ is the $|r| \times n$ matrix formed by k blocks of the form (2.1). Then $W(X : r)(t)$ is the determinant of the $|r| \times |r|$ matrix whose columns are the first $|r|$ columns of $X[\tau]$. Thus, in the case that r has only one nonzero component r_i, $W(X : r)(t)$ is a classical Wronskian.

Generalized disconjugacy. Let $\mathscr{T}$ be a class of partitions τ of points in J. The space $\mathscr{X}$ will be said to be $\mathscr{T}$-disconjugate on a subset I of J if there exists at least one $\tau \in \mathscr{T}$ such that τ is a partition of n points in I and one (and therefore all) of the following equivalent conditions D_1, D_2, D_3 holds.

D_1: (i) The dimension of $\mathscr{X}$ is at most n.

(ii) If τ is a partition of n points in I, $\tau \in \mathscr{T}$, and $c \in \mathbb{R}^n$, then there exists $x \in \mathscr{X}$ such that $x[\tau] = c$.

D_2: (i) The dimension of $\mathscr{X}$ is at least n.

(ii) If τ is a partition of n points in I, $\tau \in \mathscr{T}$, and $x[\tau] = 0$, then $x = 0$, the zero element of $\mathscr{X}$.

D_3: If τ is a partition of n points in I, $\tau \in \mathscr{T}$, and $c \in \mathbb{R}^n$, then there exists a unique $x \in \mathscr{X}$ such that $x[\tau] = c$.

If $X = [x^1, ..., x^n]$ is a basis of $\mathscr{X}$, then $\mathscr{T}$-disconjugacy of $\mathscr{X}$ on a subset I of J is equivalent to the nonsingularity of the $n \times n$ matrix $X[\tau]$ for all partitions τ of n points in I such that $\tau \in \mathscr{T}$, the set of such partitions being assumed to be nonempty. If τ is any such partition of n points in I and σ is any partition of points in J, then $x[\sigma]$ is given by the formula

$$x[\sigma] = X[\sigma](X[\tau])^{-1}\, x[\tau]. \tag{2.2}$$

In particular, with $\sigma = (t; t; ...; t)$, $x(t) = X(t)(X[\tau])^{-1}\, x[\tau]$.

This paper is primarily concerned with the cases when I is either a single point in J or a subinterval of J. Classes $\mathscr{T}$ of partitions for which results have been obtained are specified in the appropriate sections.

3. On r-Disconjugacy

Let $r=(r_1, ..., r_k) \in \mathbb{Z}^k$, $r_i \geqslant 0$, be such that $r_1 + \cdots + r_k = n$. The case of $\mathscr{T}$-disconjugacy where $\mathscr{T}$ consists of all partitions τ with $|\tau_i| = r_i$ will be denoted r-disconjugacy. When $I = \{t_0\}$, a singleton, r-disconjugacy of $\mathscr{X}$ on I means an element x of $\mathscr{X}$ is uniquely determined by specifying $D^{j-1}x_i(t_0)$, $j=1, ..., r_i$, $i=1, ..., k$. Clearly this is equivalent to the condition that

$$W(X:r)(t) \neq 0 \tag{3.1}$$

holds for any (and hence every) basis X of $\mathscr{X}$ and $t = t_0$. The space $\mathscr{X}$ is r-disconjugate on $\{t\}$ for each $t \in I$ if and only if (3.1) holds for any basis X and for each $t \in I$. When elements x of $\mathscr{X}$ are sufficiently smooth, for example if $x_i \in C^{r_i}(J)$, $i=1, ..., k$, for each $x \in \mathscr{X}$, then r-disconjugacy of $\mathscr{X}$ on each $\{t\} \subset J$ is equivalent to $\mathscr{X}$ being the solution set of a system of k linear ordinary differential equations of the form $l_i x = 0$, where

$$l_i x = \sum_{j=0}^{r_i} a_{ij} D^{r_i - j} x_i - \sum_{p \neq i} \sum_{j=1}^{r_p} b_{pj} D^{r_p - j} x_p, \qquad i=1, ..., k, \tag{3.2}$$

the coefficients a_{ij}, b_{pj} are continuous functions on J, and $a_{i0} = 1$. When $k=1$, $r=(n)$, (3.2) reduces to a linear scalar differential operator of the form (1.1). When $k=n$ and $r=(1, ..., 1)$, (3.2) defines an operator of the form $x' - Ax$ and $\mathscr{X}$ is the solution set of (1.5). Another way of writing the expression (3.2) in terms of generalized Wronskians is

$$l_i x = W(X, x : r + e^i)/W(X:r), \qquad i=1, ..., k, \tag{3.3}$$

where $X = [x^1, ..., x^n]$ is any basis of $\mathscr{X}$, $[X, x] = [x^1, ..., x^n, x]$, and $r + e^i$ is the $(n+1)$-tuple obtained by replacing r_i by $r_i + 1$ in r. When $k=1$ and $r=(n)$, (3.3) reduces to the familiar expression

$$Lu = W(u_1, ..., u_n, u)/W(u_1, ..., u_n)$$

for operators L of the form (1.1), where $(u_1, ..., u_n)$ is a linearly independent set of solutions for $Lu = 0$. In the case $k=n$, $r=(1, ..., 1)$, lx in (3.3) can be seen to be equivalent to $x' - Ax$, with $A = \Phi' \Phi^{-1}$ where Φ is a fundamental matrix for (1.5).

We now consider r-disconjugacy on I when I is a subinterval of J. When $k=1$, $r=(n)$, r-disconjugacy of $\mathscr{X}$ on I is the classical disconjugacy of the operator L of the form (1.1) for which $\mathscr{X}$, subject to obvious smoothness restrictions, is the null set. When $k=n$, $r=(1, ..., 1)$, r-disconjugacy of $\mathscr{X}$ on I is equivalent to nonoscillation on I in the sense of Schwarz and Nehari of the corresponding equation (1.5). For general k, $1 \leqslant k \leqslant n$, r-disconjugacy of $\mathscr{X}$ means that $x=0$ is the only solution of the system $lx=0$, l as in (3.2) and (3.3), such that x_i has r_i zeros in I, $i=1, ..., k$. Since so many deep but often essentially quite different results have been discovered in the extreme cases $k=1$, $k=n$, it would be of interest to investigate the structure of r-disconjugacy when $1<k<n$. While results pertinent to these cases seem to be scarce, theorems of Gingold [5] may be used to extend classical results on the boundary behaviour of systems with respect to extreme intervals of disconjugacy.

4. Criteria for $\mathscr{T}_{\pm}$-Disconjugacy and Examples

In this paper we wish to extend the idea (1.3) of a Markov system to vector valued functions and thus generalize the criterion (1.3) for classical disconjugacy to $\mathscr{T}$-disconjugacy for certain classes $\mathscr{T}$. To do this, we will impose restrictions on the cardinality $|\tau_i|$ for $\tau \in \mathscr{T}$ as well as certain order restrictions on the points in τ. We consider a prescribed vector $m=(m_1, ..., m_k) \in \mathbb{Z}^k$ such that

$$n=m_1 \geqslant m_2 \geqslant \cdots \geqslant m_k>0. \tag{4.1}$$

We also consider a prescribed set $\mathscr{R}$ of vectors $r=(r_1, ..., r_k) \in \mathbb{Z}^k$ such that

$$r_i \geqslant 0, \qquad m_i \geqslant r_i+r_{i+1}+\cdots+r_k, \qquad i=1, ..., k, \tag{4.2}$$

$$m_i e^i \in \mathscr{R} \qquad \text{and} \qquad r \in \mathscr{R}, \qquad r_i \neq 0 \Rightarrow r-e^i \in \mathscr{R}, \tag{4.3}$$

where $e^1, ..., e^k$ are the standard basis vectors in $\mathbb{R}^k$. The set $\mathscr{R}$ will be called *maximal* if it contains all r satisfying (4.2) and *minimal* if it consists only of vectors of the form $r=|r|\, e^i$, $0 \leqslant |r| \leqslant m_i$, $i=1, ..., k$. These are the largest and smallest sets $\mathscr{R}$, respectively, which satisfy (4.2) and (4.3).

m-Wronskians and primary m-Wronskians. If $x=[x^1, ..., x^n]$, $x^i \in \mathscr{X}$, we will call $W(X:r)$ an m-Wronskian if $r=(r_1, ..., r_k)$ satisfies (4.2). An m-Wronskian $W(X:r)$ will be designated primary if $r \in \mathscr{R}$. In particular, all m-Wronskians are primary if $\mathscr{R}$ is maximal while only those m-Wronskians $W(X:r)$ for which the vector r has one nonzero component are primary if $\mathscr{R}$ is minimal.

The classes $\mathscr{T}_+$, $\mathscr{T}_-$. A partition τ belongs to the class $\mathscr{T}_+$ [resp. $\mathscr{T}_-$] if $|\tau| \leqslant n$,

$$m_i \geqslant |\tau_i|+|\tau_{i+1}|+\cdots+|\tau_k|, \qquad i=1, ..., k, \tag{4.4}$$

$$(|\tau_1(t)|, ..., |\tau_k(t)|) \in \mathscr{R}, \qquad \text{for each} \quad t \in \tau, \tag{4.5}$$

and

$$t \in \tau_i, \qquad s \in \tau_j, \qquad i < j \Rightarrow t \leqslant s \text{ [resp. } t \geqslant s]. \tag{4.6}$$

We emphasize the dependence of the classes $\mathscr{T}_\pm$ on the prescribed vector m, the set $\mathscr{R}$, and the suffix $+$ or $-$. The vector m restricts the cardinality of the components of a partition $\tau \in \mathscr{T}_\pm$; the set $\mathscr{R}$ restricts the multiplicities of occurrences in τ_i of an entry t common to two or more components of τ and the suffix denotes an order on τ. The set $\mathscr{R}$, in particular, has been introduced to allow situations where various components of the value function are not necessarily independent of each other as occurs in some interesting applications.

The signum functions θ_+, θ_-. A function θ_+ [resp. θ_-] whose domain is the set of all k-tuples $r = (r_1, ..., r_k)$ of integers r_i satisfying (4.2) and which has values ± 1 is a signum function corresponding to $\mathscr{T}_+$ [resp. $\mathscr{T}_-$] if there exist functions φ_i: $\{0, 1, ..., m_i\} \to \{-1, 1\}$, $i = 1, ..., k$, such that $\varphi_i(0) = 1$ and

$$\theta_+(r) = \varphi_1(r_1 + \cdots + r_k) \prod_{i=2}^{k} \varphi_{i-1}(r_i + \cdots + r_k)\, \varphi_i(r_i + \cdots + r_k)$$

$$\left[\text{resp. } \theta_-(r) = \varphi_1(r_1 + \cdots + r_k) \prod_{i=2}^{k} \varphi_{i-1}(r_i + \cdots + r_k) \right.$$

$$\left. \times\ \varphi_i(r_i + \cdots + r_k)(-1)^{r_{i-1}(r_i + \cdots + r_k)} \right]. \tag{4.7}$$

Vector Markov systems. An n-tuple $X = [x^1, ..., x^n]$ of elements $x^i \in \mathscr{X}$ is a $\mathscr{T}_+$-Markov system [resp. $\mathscr{T}_-$-Markov system] on an interval I if there exists a signum function θ_+ [resp. θ_-] such that

$$\theta_+(r)\, W(X : r)(t) \geqslant 0 \text{ [resp. } \theta_-(r)\, W(X : r)(t) \geqslant 0] \tag{4.8}$$

holds for each $t \in I$ and all m-Wronskians $W(X : r)$, with strict inequality holding for each $t \in I$ if $W(X : r)$ is a primary m-Wronskian.

We see that the role of the functions $\theta_\pm$ is that $\theta_\pm(r)$ specifies the sign of the corresponding m-Wronskian $W(X : r)$ for a $\mathscr{T}_\pm$-Markov system X. Expressions (4.7) show how the signs of the general m-Wronskians must be related to those of the m-Wronskians $W(X : r)$ in which r has exactly one nonzero component, since $\theta_\pm(qe^i) = \varphi_i(q)$ is, by (4.8), the sign of the leading $q \times q$ minor of the matrix (2.1). The signs of these minors determine the signs of all other m-Wronskians in a Markov system by (4.7), (4.8).

THEOREM 4.1. *Let $\mathscr{X}$ have dimension n and let I be a subinterval of J.*

Suppose there exist $x^i \in \mathscr{X}$, $i=1, ..., n$, such that $X=[x^1, ..., x^n]$ is a $\mathscr{T}_+$ [resp. $\mathscr{T}_-$]-Markov system on I. Then $\mathscr{X}$ is $\mathscr{T}_+$ [resp. $\mathscr{T}_-$]-disconjugate on I.

The conditions of Theorem 4.1 are not in general necessary for $\mathscr{T}_+$-disconjugacy of $\mathscr{X}$ on I without some additional restrictions on $\mathscr{X}$ and I. The most important of these is the following restriction on $\mathscr{X}$: If x_1 has a zero of multiplicity m_1-h at t_0, then the component x_i has a zero of multiplicity m_i-h at t_0 for $1 \leqslant h \leqslant m_i - 1$ and $i<k$. Specifically,

$$\begin{gathered} x \in \mathscr{X}, \qquad D^{j-1}x_1(t_0)=0, \qquad j=1, ..., m_1-h \Rightarrow D^{j-1}x_i(t_0)=0, \\ j=1, ..., m_i-h, \text{ if } i<k \text{ and } 1 \leqslant h \leqslant m_i-1. \end{gathered} \tag{4.9}$$

Condition (4.9) is satisfied at all $t_0 \in J$ when $x_i = l_i x_1$, $i=2, ..., k-1$, for each $x \in \mathscr{X}$, where l_i is a linear ordinary differential operator of order not exceeding $m_1 - m_i = n - m_i$. The condition holds trivially for $k \leqslant 2$ since it only pertains to those components x_i of x such that $2 \leqslant i \leqslant k-1$, if any.

THEOREM 4.2. *Let I be a closed subinterval of J. Suppose that $\mathscr{X}$ is $\mathscr{T}_+$ [resp. $\mathscr{T}_-$]-disconjugate on I and that it satisfies* (4.9), *where t_0 is the left [resp. right] endpoint of I. Then $\mathscr{X}$ has a basis $X=[x^1, ..., x^n]$ which is a $\mathscr{T}_+$ [resp. $\mathscr{T}_-$]-Markov system on $I \setminus \{t_0\}$.*

When $\mathscr{T}_\pm$-disconjugacy of $\mathscr{X}$ on a closed interval I implies $\mathscr{T}_\pm$-disconjugacy on a larger interval containing t_0 in its interior, then Theorem 4.2 applied to a slightly larger interval shows that the exclusion of the endpoint t_0 from the domain of the Markov system in that theorem is unnecessary. This is the case for classical disconjugacy [1, p. 94]. It is also the case for $\mathscr{T}_\pm$-disconjugacy when the set $\mathscr{R}$ is maximal. Even in some situations when $\mathscr{R}$ is not maximal, as in the case of disfocality [15], the point t_0 need not be excluded for this reason. However, it is not true in general that $\mathscr{T}_\pm$-disconjugacy on a closed interval I implies the same property on such a larger interval, as shown by Example 4.6.

To place Theorems 4.1, 4.2 in the context of ordinary differential equations, observe that the space $\mathscr{X}_1 = \{x_1 : x \in \mathscr{X}\}$ is (n)-disconjugate on I if $\mathscr{X}$ is $\mathscr{T}_+$ [resp. $\mathscr{T}_-$]-disconjugate on I, by (4.1), (4.4). Thus, subject to a technical smoothness requirement, $\mathscr{X}_1$ is the solution set of a linear scalar differential equation, $Lu=0$, where L is of the form (1.1) and is disconjugate on I as discussed in Section 3. Moreover, from (2.2), for each $x \in \mathscr{X}$, we have $x=(x_1, ..., x_k)^T = (u, l_2 u, ..., l_k u)^T$, $Lu=0$, where l_i is a linear function from $C^n(I)$ to $C^{m_i-1}(I)$ and a solution u of $Lu=0$ is determined uniquely by the value of $(u, l_2 u, ..., l_k u)^T$ at any partition $\tau \in \mathscr{T}_+$ [resp. $\mathscr{T}_-$] such that $|\tau| = n$. Thus Theorem 4.1 gives sufficient conditions that these families of multipoint boundary value problems associated with $Lu=0$

have unique solutions. Theorem 4.2 shows that the conditions are also necessary in quite general circumstances.

The following examples are obtained by special choices of the functions $l_2, ..., l_k$.

EXAMPLE 4.3. Let $\mathscr{X} = \{(u, Du, ..., D^{n-1}u)^T: Lu = 0\}$, where L is a linear differential operator of the form (1.1). Then L is right-disfocal [resp. left-disfocal] on I if and only if $\mathscr{X}$ is $\mathscr{T}_+$ [resp. $\mathscr{T}_-$]-disconjugate on I where $m = (n, n-1, ..., 1)$ is the k-tuple $(k = n)$ defining $\mathscr{T}_\pm$ in (4.4) and $\mathscr{R}$ consists of these n-tuples $r = (r_1, ..., r_n)$ of integers $r_i \geqslant 0$ such that $n - i + 1 \geqslant r_i + \cdots + r_n$ and, if $0 < r_i$, $j < i$, then $r_j \leqslant i - j$. This condition ensures that if $\tau \in \mathscr{T}_\pm$ and $t \in \tau_i$, then the multiplicity of $t \in \tau_j$, $j < i$, is such that the entry $D^{j-1}u(t)$ does not occur more than once in the value vector $(u, Du, ..., D^{n-1}u)[\tau]$. In this case the condition that $\mathscr{X}$ has a $\mathscr{T}_+$-Markov system, with $\theta_+(r) = 1$ for all r, is sufficient for right-disfocality of L on any interval I and necessary for right-disfocality if I is closed. This is equivalent to condition (1.4) although (1.4) involves fewer Wronskians than the definition of a $\mathscr{T}_+$-Markov system. These inequalities imply the remaining inequalities in (4.8) in this case.

An obvious generalization of disfocality, with necessary and sufficient conditions, is obtained by replacing the operator D^{j-1} by a general linear differential operator of order $j-1$. We are not of course restricted to differential operators.

EXAMPLE 4.4. Let $\mathscr{X} = \{(u, u \circ f)^T: L_2 u = 0\}$, with L_2 a second order linear ordinary differential operator on $C^2(J)$, $f: J \to J$, $m = (2, 1)$, and $\mathscr{R}$ maximal. Then $\mathscr{X}$ is $\mathscr{T}_+$ [resp. $\mathscr{T}_-$]-disconjugate on an interval I provided

$$L_2 u = 0,\ t_1, t_2 \in I,\ t_1 \neq t_2,\ u(t_1) = u(t_2) = 0 \Rightarrow u = 0$$

and

$$L_2 u = 0,\ t_1, t_2 \in I,\ t_1 \leqslant t_2 \text{ [resp. } t_1 \geqslant t_2],\ u(t_1) = u(f(t_2)) = 0 \Rightarrow u = 0.$$

Thus $\mathscr{T}_\pm$-disconjugacy of $\mathscr{X}$ on I means not only disconjugacy of L on I but also places a restriction, depending on f, on the location of further zeros of solutions u of $L_2 u = 0$ which have exactly one zero in I. It is clear that this approach may be used to investigate the null set of L outside of intervals of disconjugacy.

For any interval $I \subset J$, a sufficient condition that $\mathscr{X}$ be $\mathscr{T}_+$ [resp. $\mathscr{T}_-$]-disconjugate on I is that there exist solutions u_1, u_2 of $L_2 u = 0$ such that

$$u_1(u_1 Du_2 - u_2 Du_1) > 0, \qquad (u_1 \circ f)(u_1(u_2 \circ f) - u_2(u_1 \circ f)) > 0 \text{ [resp. } < 0]$$

hold on I. To see this, observe that if $X=[x^1, x^2]$ with $x^j=(u_j, u_j\circ f))^T$, $j=1, 2$, then

$$W(X:1,0)=u_1, \qquad W(X:2,0)=u_1Du_2-u_2Du_1$$
$$W(X:0,1)=u_1\circ f, \qquad W(X:1,1)=u_1(u_2\circ f)-u_2(u_1\circ f).$$

All of these Wronskians must be nonzero if X is a $\mathscr{T}_\pm$-Markov system. Thus the first two may be assumed to be positive so that

$$\theta_\pm(1,0)=\varphi_1(1)=1, \qquad \theta_\pm(2,0)=\varphi_1(2)=1.$$

The signs of the remaining determinants are then, by (4.7),

$$\theta_\pm(0,1)=\varphi_2(1), \qquad \theta_\pm(1,1)=\varphi_1(2)\,\varphi_1(1)\,\varphi_2(1)(\pm 1)=\varphi_2(1)(\pm 1).$$

Here, and throughout this paper, whenever the alternative "$\pm$" is given, one sign must be chosen consistently throughout the expression. Thus, if the last two Wronskians have the same sign [resp. opposite signs], we can infer $\mathscr{T}_+$ [resp. $\mathscr{T}_-$]-disconjugacy of $\mathscr{X}$ on I. If $f(t)\neq t$ for each $t\in I$ and I is closed, then these conditions are also necessary for $\mathscr{T}_\pm$-disconjugacy of $\mathscr{X}$ on I.

EXAMPLE 4.5. Let $\mathscr{X}=\{(x_1,x_2)^T: D^2x_1=0, \quad x_2(t)=\int_t^{t+1}x_1\}$ with $m=(2,1)$ and $\mathscr{R}$ maximal. Clearly $x_1(t)=c_1+c_2t$, $x_2(t)=c_1+c_2(t+\frac{1}{2})$, where c_1, c_2 are arbitrary constants. Then $\mathscr{X}$ is $\mathscr{T}_+$ [resp. $\mathscr{T}_-$]-disconjugate on I if and only if the determinants

$$\begin{vmatrix} 1 & t_1 \\ 1 & t_2 \end{vmatrix}=t_2-t_1, \qquad \begin{vmatrix} 1 & t_1 \\ 1 & t_2+\frac{1}{2} \end{vmatrix}=t_2-t_1+\tfrac{1}{2}, \qquad t_1, t_2\in I$$

are nonzero when $t_1\neq t_2$, $t_1\leqslant t_2$ [resp. $t_1\geqslant t_2$], respectively. Thus $\mathscr{X}$ is $\mathscr{T}_+$-disconjugate on every real interval I. The situation is changed radically when we consider $\mathscr{T}_-$-disconjugacy: $\mathscr{X}$ is $\mathscr{T}_-$-disconjugate on I if and only if the length of I is less than $\frac{1}{2}$ or the length of I is equal to $\frac{1}{2}$ and I is not closed. For example, $\mathscr{X}$ is not $\mathscr{T}_-$-disconjugate on $[-\frac{1}{2}, 0]$ since, if $x(t)=(t, t+\frac{1}{2})^T$, then $x\in\mathscr{X}$ and $x[0; -\frac{1}{2}]=0$; however, $\mathscr{X}$ is $\mathscr{T}_-$-disconjugate on every proper subinterval of $[-\frac{1}{2}, 0]$.

If we choose $X=[x^1, x^2]$, where $x^1(t)=(1,1)^T$ and $x^2(t)=(t, t+\frac{1}{2})^T$, we find

$$W(X:1,0)(t)=x_1^1(t)=1, \qquad W(X:2,0)(t)=x_1^1(t)\,Dx_1^2(t)-x_1^2(t)\,Dx_1^1(t)=1,$$
$$W(X:0,1)(t)=x_2^1(t)=1, \qquad W(X:1,1)(t)=x_1^1(t)\,x_2^2(t)-x_1^2(t)\,x_2^1(t)=\tfrac{1}{2}$$

which are all primary Wronskians and positive for all $t\in\mathbb{R}$. Here

$\varphi_1(1)=\varphi_1(2)=\varphi_2(1)=1$ so that, from (4.7), $\theta_+(1,0)=\theta_+(2,0)=\theta_+(0,1)=\theta_+(1,1)=1$ and the conditions and conclusion of Theorem 4.1 are verified for $\mathscr{T}_+$-disconjugacy.

To consider $\mathscr{T}_-$-disconjugacy of $\mathscr{X}$ in Example 4.5 on the interval $(-\frac{1}{2}, 0)$, let $X=[x^1, x^2]$ with $x^1(t)=(t, t+\frac{1}{2})^T$ and $x^2(t)=(1,1)^T$ so that the four Wronskians are

$$W(X:1,0)(t)=t, \qquad W(X:2,0)(t)=-1$$
$$W(X:0,1)(t)=t+\tfrac{1}{2}, \qquad W(X:1,1)(t)=-\tfrac{1}{2}.$$

Then $\varphi_1(1)=\varphi_1(2)=-1$, $\varphi_2(1)=1$ give, from (4.7),

$$\theta_-(1,0)=-1, \qquad \theta_-(2,0)=-1, \qquad \theta_-(0,1)=1, \qquad \theta_-(1,1)=-1$$

and the condition (4.8) that X be a $\mathscr{T}_-$-Markov system is satisfied if $-\frac{1}{2}<t\leqslant 0$.

EXAMPLE 4.6. Let $\mathscr{X}=\{(x_1,x_2)^T\colon D^2x_1=0,\ \ x_2(t)=tx_1(t)\}$ and let $m=(2,1)$ with $\mathscr{R}$ minimal. Then $\mathscr{X}$ is $\mathscr{T}_+$-disconjugate on every closed interval $[0,b]$ but it is not $\mathscr{T}_+$-disconjugate on any interval $[a,b]$ if $a<0<b$.

5. PROOFS

Two determinant identities presented here play a key role in this paper. Let $A=[a_i^j]$ be any $n\times n$ matrix and let $a_{r_1\cdots r_m}^{s_1\cdots s_m}$ denote its minor determined by the rows $r_1,\ldots,r_m$ and the columns $s_1,\ldots,s_m$. Further, let $a(i,\lambda)_{r_1\cdots r_m}^{s_1\cdots s_m}$ denote the corresponding minor for the matrix obtained by replacing the ith row of A by $\lambda=[\lambda^1,\ldots,\lambda^n]$. Then, if $i, j\in(1,\ldots,n)$, $i\neq j$,

$$a(i,\lambda)_{1\cdots\hat{j}\cdots n}^{1\cdots n-1}\, a_{12\cdots n}^{12\cdots n}=a_{1\cdots\hat{j}\cdots n}^{1\cdots n-1}\, a(i,\lambda)_{12\cdots n}^{12\cdots n}+(-1)^{i-j+1}\, a_{1\cdots\hat{i}\cdots n}^{1\cdots n-1}\, a(j,\lambda)_{12\cdots n}^{12\cdots n}, \tag{5.1}$$

where $(1,\ldots,\hat{j},\ldots,n)$ denotes the $(n-1)$-tuple obtained by omitting j from $(1,\ldots,n)$. Also, if $1\leqslant p\leqslant n-1$ and

$$b_i^j=a_{12\cdots p,\,p+i}^{12\cdots p,\,p+j}, \qquad i,j=1,\ldots,n-p,$$

then

$$a_{12\cdots n}^{12\cdots n}\left(a_{12\cdots p}^{12\cdots p}\right)^{n-p-1}=b_{12\cdots n-p}^{12\cdots n-p}. \tag{5.2}$$

Expression (5.2) is Sylvester's identity (cf. [4, p. 32]). Equation (5.1) is essentially proved as Lemma 2 in [15], but the following proof is much simpler. It suffices to prove (5.1) when $rkA = n$ since the general result then follows by continuity of the determinants involved as functions of the entries in A and λ. Both sides of (5.1) are linear real-valued functions of λ and it is simple to check that both sides are equal if $\lambda = (a_p^1, ..., a_p^n)$, $p = 1, ..., n$. Thus they are equal for all λ since these vectors span $\mathbb{R}^n$.

We consider partitions which have the special form shown in the displays $(5.3)_\pm$,

$$\tau = (\overbrace{t_1, ..., t_p}^{p}, \overbrace{t, ..., t}^{q}; \overbrace{t, ..., t}^{r_2}; ...; \overbrace{t, ..., t}^{r_k})$$

$$\sigma = (\overbrace{t_1, ..., t_{p-1}}^{p-1}, s, \overbrace{t, ..., t}^{q}; \overbrace{t, ..., t}^{r_2}; ...; \overbrace{t, ..., t}^{r_k})$$

$$\rho = (\overbrace{t_1, ..., t_p}^{p}, s, \overbrace{t, ..., t}^{q}; \overbrace{t, ..., t}^{r_2}; ...; \overbrace{t, ..., t}^{r_k - 1})$$

$$t_p < s \leqslant t, \qquad p + q = r_1 \tag{5.3$_+$}$$

$$\tau = (\overbrace{t, ..., t}^{q}, \overbrace{t_1, ..., t_p}^{p}; \overbrace{t, ..., t}^{r_2}; ...; \overbrace{t, ..., t}^{r_k})$$

$$\sigma = (\overbrace{t, ..., t}^{q}, s, \overbrace{t_2, ..., t_p}^{p-1}; \overbrace{t, ..., t}^{r_2}; ...; \overbrace{t, ..., t}^{r_k})$$

$$\rho = (\overbrace{t, ..., t}^{q}, s, \overbrace{t_1, ..., t_p}^{p}; \overbrace{t, ..., t}^{r_2}; ...; \overbrace{t, ..., t}^{r_k - 1})$$

$$t \leqslant s < t_1, \qquad p + q = r_1. \tag{5.3$_-$}$$

The number of terms in each grouping is indicated by the corresponding brace. Thus, for example, $|\tau_1(t)| = q$, $|\tau_1| = r_1 = p + q$, τ_i contains at most one distinct entry t if $i \geqslant 2$ and $|\tau_i| = |\tau_i(t)| = r_i$. Note that $|\tau| = |\sigma| = |\rho| = r_1 + \cdots + r_k$.

LEMMA 5.1. *Let* $X = [x^1, ..., x^n]$, $x^j \in \mathscr{X}$, $j = 1, ..., n$, *and let* τ, ρ, σ *be as in* $(5.3)_\pm$. *Then*

$$x[\sigma]_{12\cdots n-1}^{12\cdots n-1}\, x[\tau]_{12\cdots n}^{12\cdots n} = x[\tau]_{12\cdots n-1}^{12\cdots n-1}\, x[\sigma]_{12\cdots n}^{12\cdots n} + (\pm 1)\, x[\tau]_{1\cdots \hat{p}\cdots n}^{1\cdots n-1}\, x[\rho]_{12\cdots n}^{12\cdots n}. \tag{5.4$_\pm$}$$

This lemma follows from identity (5.1). We first prove the case $(5.4)_+$ when $s < t$. With $A = X[\tau]$ and $i = p$, $j = n$, $\lambda = [x_1^1(s), ..., x_1^n(s)]$, we find, if we observe that $(-1)^{i-j+1} a(j, \lambda)_{12\cdots n}^{12\cdots n} = x[\rho]_{12\cdots n}^{12\cdots n}$, $(5.4)_+$ holds if τ, σ, and ρ are as given in $(5.3)_+$. To extend this to the case $s = t$, differentiate the relationship $(5.4)_+$ now established for $s<t$, q times with respect to s; the limit, $s \to t-$, of the resulting expression gives $(5.4)_+$ for $s=t$.

The proof of $(5.4)_-$ is similar. With $A=X[\tau]$, $i=q+|\tau_1(t_1)|$, $j=n$, $\lambda=[x^1(s), ..., x^n(s)]$, and τ, σ, ρ as in $(5.3)_-$ we again find $(5.4)_-$, the only change being that the determinants $x[\sigma]_{12\cdots n-1}^{12\cdots n-1}$, $x[\sigma]_{12\cdots n}^{12\cdots n}$, $x[\rho]_{12\cdots n}^{12\cdots n}$ are multiplied by $(-1)^{|\tau_1(t_1)|-1}$, $(-1)^{|\tau_1(t_1)|-1}$, $(-1)^{|\tau_1(t_1)|}$, respectively, which gives rise to the term (-1) in the expression on the right of $(5.4)_-$. The case $s=t$ is established by a limit process as before.

PROPOSITION 5.2. *Let $X=[x^1, ..., x^n]$ be a $\mathscr{T}_\pm$-Markov system on an interval I. Then the inequality*

$$\theta_\pm(|\tau_1|, ..., |\tau_k|)\, x[\tau]_{12\cdots h}^{12\cdots h} \geqslant 0, \qquad h=|\tau| \leqslant n \tag{5.5$_\pm$}$$

holds for all partitions τ of points in I satisfying (4.4) *and* (4.6). *Also the inequality* $(5.5)_\pm$ *holds strictly if $\tau \in \mathscr{T}_\pm$.*

This proposition implies Theorem 4.1 since, under the hypothesis there, X is a basis of $\mathscr{X}$ and det $X[\tau] \neq 0$ if $|\tau|=n$, $\tau \in \mathscr{T}_\pm$, means that the zero of $\mathscr{X}$ is the only element which is zero at τ.

Proof of Proposition 5.2. We order the vectors $(|\tau_1|, ..., |\tau_k|)$ satisfying (4.4) lexicographically and furnish a proof of Proposition 5.2 by an induction argument on $|\tau|$ and $(|\tau_1|, ..., |\tau_k|)$. Since several subsidiary inductive hypotheses are needed throughout the proof, we shall refer to the following as the basic induction assumptions:

(A_1) The proposition holds if $|\tau|<n$.

(A_2) The proposition holds if $|\tau|=n$ and $|\tau_k|<r_k$ for some r_k, $1 \leqslant r_k \leqslant m_k$.

Assumption (A_1) holds trivially for $n=2$ without restriction on k. Assumption (A_2) holds for $k=2$, $r_k=1$ without restriction on n since, in that case, the proposition is essentially (1.3) and Theorem V of Pólya [20]. Our proof will show that the hypothesis of the proposition with (A_1) and (A_2) implies that the result holds for $|\tau|=n$ and $|\tau_k|=r_k$ and thus, by induction, for $|\tau_k| \leqslant m_k$. Now, if X has more than k rows, we find that the basic induction assumptions are verified with k replaced by $k+1$ and $r_{k+1}=1$, so that the proposition holds by induction for arbitrary n and k. The proof is presented in three steps.

STEP 1. *Inequality* (5.5) *holds if the partition τ of points in I satisfies* (4.4) *and* (4.6) *and has the special form given in* $(5.3)_\pm$.

By our basic induction assumption, the assertion of Step 1 holds if

$|\tau| < n$ and, when $|\tau| = n$, it is true if $|\tau_k| < r_k$. It remains only to show that this implies that it holds if $|\tau| = n$ and $|\tau_k| = r_k$ and hence in full generality, by induction. This is shown by a further induction argument on $|\tau_1| - |\tau_1(t)|$, the number, counting multiplicities, of entries in τ_1 which are distinct from t. Assume that the assertion of Step 1 holds if $|\tau_1| - |\tau_1(t)| < p$, $1 \leqslant p \leqslant n$. It is clear that this is true for $p = 1$ since, in that case,

$$\det X[\tau] = W(X\colon |\tau_1|, ..., |\tau_k|)(t),$$

a Wronskian determinant which, by hypothesis, satisfies (4.8) and therefore (5.5). It remains to show that this implies that Step 1 holds if $|\tau_1| - |\tau_1(t)| = p$, and hence for $0 \leqslant |\tau_1| - |\tau_1(t)| \leqslant n$.

Consider the identity $(5.4)_\pm$ with $s = t$ in $(5.3)_\pm$. By our induction assumption on p, since $|\sigma_1| - |\sigma_1(t)| = p - 1$, and by our basic induction assumption (A_2), since $|\rho_k| = r_k - 1$,

$$\theta_\pm(r_1, ..., r_k)\, x[\sigma]_{12\cdots n}^{12\cdots n} \geqslant 0, \qquad \theta_\pm(r_1 + 1, r_2, ..., r_k - 1)\, x[\rho]_{12\cdots n}^{12\cdots n} \geqslant 0. \tag{5.6}$$

The induction assumption (A_1) implies

$$\theta_\pm(r_1, r_2, ..., r_k - 1)\, x[\sigma]_{12\cdots n-1}^{12\cdots n-1} \geqslant 0, \quad \theta_\pm(r_1, r_2, ..., r_k - 1)\, x[\tau]_{12\cdots n-1}^{12\cdots n-1} \geqslant 0$$
$$\theta_\pm(r_1 - 1, r_2, ..., r_k)\, x[\tau]_{12\cdots \hat{p} \cdots n}^{12\cdots n-1} \geqslant 0. \tag{5.7}$$

From (4.7), the definition of $\theta_\pm$, it can be checked that

$$\begin{aligned}\theta_\pm&(r_1, r_2, ..., r_k - 1)\, \theta_\pm(r_1, ..., r_k)\\ &= (\pm 1)\, \theta_\pm(r_1 - 1, r_2, ..., r_k)\, \theta_\pm(r_1 + 1, r_2, ..., r_k - 1)\end{aligned}$$

and hence, from $(5.4)_\pm$, (5.6), and (5.7),

$$\theta_\pm(r_1, ..., r_k)\, x[\tau]_{12\cdots n}^{12\cdots n} \geqslant 0$$

if $x[\sigma]_{12\cdots n-1}^{12\cdots n-1} \neq 0$, completing the proof that Step 1 holds if $|\tau_1| - |\tau_1(t)| = p$. When $x[\sigma]_{12\cdots n-1}^{12\cdots n-1} = 0$, we use the following perturbation argument. In $(5.3)_\pm$, for all $i < k$, replace each t in τ_{i+1} by $t \pm \sum_{j=1}^{i} 2^{-j}\varepsilon$, $\varepsilon > 0$, and make the corresponding modification in σ and ρ with $s = t$ as before. Thus modified, all of the partitions in $(5.3)_\pm$ belong to $\mathscr{T}_\pm$, since they satisfy not only (4.4) and (4.6) but also (4.5) as τ_i and τ_j have no entries in common if $i < j$. Now, by the basic induction assumptions, the inequalities in (5.6), (5.7) are all strict with the possible exception of that for $x[\sigma]_{12\cdots n}^{12\cdots n}$ in (5.6) which might now fail since $|\sigma_k| = r_k$ and the requirement of (A_2) does not hold. However, the cofactor of $x[\sigma]_{12\cdots n-1}^{12\cdots n-1}$

in $x[\sigma]_{12\cdots n}^{12\cdots n}$ is $D^{r_k-1}x_k^n$ and the only other occurrence of this term in $(5.4)_\pm$ is in $x[\tau]_{12\cdots n}^{12\cdots n}$ as the cofactor of $x[\tau]_{12\cdots n-1}^{12\cdots n-1}$. Thus, we may replace $D^{r_k-1}x_k^n$ by $D^{r_k-1}x_k^n+\Delta$ without altering the equality in $(5.4)_\pm$ and without altering any of the determinants in (5.6), (5.7) other than $x[\sigma]_{12\cdots n}^{12\cdots n}$. Further, Δ may be chosen so that $\theta_\pm(r_1, ..., r_k)\, x[\sigma]_{12\cdots n}^{12\cdots n}>0$. Thus modified, $X[\tau]$ satisfies $\theta_\pm(r_1, ..., r_k)\, x[\tau]_{12\cdots n}^{12\cdots n}>0$. Moreover, since $\theta_\pm(r_1, ..., r_k)\, x[\sigma]_{12\cdots n}^{12\cdots n}\geqslant 0$ if $\varepsilon=\Delta=0$, we may have Δ as small as we please provided ε is sufficiently small. By considering $\varepsilon\to 0+$, $\Delta\to 0$ we find that $(5.5)_\pm$ holds in this case also.

STEP 2. *If $\tau\in\mathscr{T}_\pm$, and τ is a partition of points in I of the form given in* $(5.3)_\pm$, *then* $(5.5)_\pm$ *holds with strict inequality.*

Here we use another inductive argument on p. By our basic induction assumption, Step 2 holds if $|\tau|<n$. Now assume it holds if $|\tau|=n$ and $|\tau_1|-|\tau_1(t)|>p$; this is true for $p=n-1$, by Theorem V of Pólya [20]. Consider $(5.4)_+$ [resp. $(5.4)_-$] with $t_p<s<t$ in $(5.3)_+$ [resp. $t<s<t_1$ in $(5.3)_-$], so that $|\tau_1|-|\tau_1(t)|=p$. Then $\tau\in\mathscr{T}_+$ [resp. $\mathscr{T}_-$] is equivalent to $(q, r_2, ..., r_k)\in\mathscr{R}$. It was established in Step 1 that the nonstrict inequality (5.5) is satisfied and hence (5.6) and (5.7) hold for the partitions in $(5.3)_+$ [resp. $(5.3)_-$]. The inequality satisfied by $x[\rho]_{12\cdots n}^{12\cdots n}$ in (5.6) is strict since $(q, r_2, ..., r_k)\in\mathscr{R}$ implies $(q, r_2, ..., r_k-1)\in\mathscr{R}$, by (4.3) and since $|\rho|=n$, $|\rho_1|-|\rho_1(t)|=p+1$. The inequalities (5.7) are all strict by (A_1) since they pertain to the $\mathscr{T}_\pm$-Markov system $[x^1, ..., x^{n-1}]$ and all the partitions involved are in $\mathscr{T}_\pm$. Thus the strict inequality $(5.5)_\pm$ follows from $(5.4)_\pm$ when $|\tau_1|-|\tau_1(t)|=p$ and Step 2 holds by backwards induction on p.

STEP 3. *The preceding two steps, together with the basic induction assumption* (A_1), *imply that the conclusion of Proposition* 5.2 *holds for* $|\tau|=n$, *completing the proof of the proposition.*

We first prove this statement for the case of $\mathscr{T}_+$-disconjugacy. Define $y_i^j\colon J\to\mathbb{R}$ by

$$y_i^j(t)=x[\tau]_{12\cdots p,p+i}^{12\cdots p,p+j},\qquad i=1, ..., k,\quad j=1, ..., n-p, \tag{5.8}$$

where τ has the special form $(5.3)_+$ with the additional restriction that $|\tau_i(t)|=1$, $i=1, ..., k$. For any partition τ of points in J with $|\tau_1|\geqslant p$, Sylvester's identity (5.2) implies, if $p<h\leqslant|\tau|$,

$$x[\tau]_{12\cdots h}^{12\cdots h}\,(x[\tau]_{12\cdots p}^{12\cdots p})^{h-p-1}=y[\beta_p\tau]_{12\cdots h-p}^{12\cdots h-p}, \tag{5.9}$$

if $\beta_p\tau$ denotes the partition obtained by deleting the first p points from τ, since $y[\beta_p\tau]_i^j=y_i^j(t)=x[\tau]_{12\cdots p,p+i}^{12\cdots p,p+j}$. We now consider (5.9) when τ has the special form $(5.3)_+$. Since $(j, 0, ..., 0)\in\mathscr{R}$, $j=1, ..., n$, it follows that

$W(X:j, 0, ..., 0) \neq 0$ and it may be assumed that $W(X:p, 0, ..., 0) > 0$ or, equivalently, that $\theta_+(p, 0, ..., 0) = \varphi_1(p) = 1$. This can always be achieved by replacing the column x^p of X by $-x^p$ if necessary. By Theorem V of Pólya [20], it now follows that $x[\tau]_{12\cdots p}^{12\cdots p} > 0$ if $|\tau_1| \geqslant p$ and therefore, from (5.9) $x[\tau]_{12\cdots h}^{12\cdots h}$ and $y[\beta_p\tau]_{12\cdots h-p}^{12\cdots h-p}$ are both of the same sign or both zero, $p < h \leqslant n$. Therefore, from Steps 1, 2,

$$\theta_+(|\tau_1|, ..., |\tau_k|)\, y[\beta_p\tau]_{12\cdots h-p}^{12\cdots h-p} \geqslant 0, \qquad h = |\tau| \tag{5.10}$$

if τ has the form $(5.3)_+$ and satisfies (4.4), (4.6), and strict inequality holds if $\tau \in \mathscr{T}_+$. But, in this case, $y[\beta_p\tau]_{12\cdots h-p}^{12\cdots h-p} = W(Y:q, r_2, ..., r_k)(t)$, a Wronskian determinant associated with the matrix $Y = [y^1, ..., y^{n-p}]$ defined by (5.8). Thus the preceding assertion about (5.10) is the condition that Y is a $\hat{\mathscr{T}}_+$-Markov system on $\hat{I} = \{t \in I: t > t_p\}$ where $\hat{\mathscr{T}}_+$ is

$$\hat{m}_i = \min\{n-p, m_i\}, \quad i = 1, ..., k, \qquad \hat{\mathscr{R}} = \{\hat{r} : \hat{r} + pe^1 \in \mathscr{R}\},$$

and $\hat{\theta}_+(q, r_2, ..., r_k) = \theta_+(p+q, r_2, ..., r_k)$. Since $n - p < n$, we therefore conclude from the basic induction hypothesis (A_1) that (5.10) is satisfied if τ is any partition of $h \leqslant n$ points in I with $|\tau_1| \geqslant p$, and $\beta_p\tau$ are points in $\hat{I}$ such that τ satisfies (4.4), (4.6) with strict inequality if $\tau \in \mathscr{T}_+$. By (5.9), this establishes the assertion of Proposition 5.2 for these partitions τ.

By appropriate choices of p, we have now proved Proposition 5.2 except when $|\tau_i(t_1)| > 0$ for some $i > 1$. This case can be handled as before. First observe that if the semicolons in $(5.3)_+$ are moved to allow some of the points $(t_1, ..., t_p)$ in τ_i for $i > 1$ we can still conclude the validity of the assertion of Step 1 for these τ, provided the corresponding changes are made in σ and ρ at the appropriate parts of the proof. The remaining steps may also be completed in this way. A minor modification is needed in that we use the general form of the induction hypothesis (A_1) rather than the special case of Pólya's Theorem V to prove the positivity of $x[\tau]_{12\cdots p}^{12\cdots p}$ in (5.9).

The proof of Step 3 for $\mathscr{T}_-$-disconjugacy may also be carried out in this way or established from the foregoing by the change of variables $t \to -t$.

Proof of Theorem 4.2. Let $\mathscr{X}$ be $\mathscr{T}_+$ [resp. $\mathscr{T}_-$]-disconjugate on the closed interval I and let t_0 be the left [resp. right] endpoint of I. Then $\mathscr{X}$ has dimension n. We let $X = [x^1, ..., x^n]$ be a basis of $\mathscr{X}$ satisfying $D^{i-1}x_1^j(t_0) = 0$, $D^{n-j}x_1^j(t_0) \neq 0$, $i = 1, ..., n-j$, $j = 1, ..., n$. Then, for every $\tau \in \mathscr{T}_+$ [resp. $\mathscr{T}_-$], $x[\tau]_{12\cdots h}^{12\cdots h} \neq 0$ if $|\tau| = h < n$ and τ is a partition of points in $I \setminus \{t_0\}$ or if $|\tau| = h = n$ and τ is a partition of points in I. This is the case since, if $x[\tau]_{12\cdots h}^{12\cdots h} = 0$, $|\tau| = h \leqslant n$, then there exists $c \in \mathbb{R}^h$ such that, for $x = c_1x^1 + \cdots + c_hx^h$, x has a zero of multiplicity $n-h$ at t_0 and x is zero at τ. Since $n - h + |\tau| = n$, this contradicts the disconjugacy of $\mathscr{X}$. The sign of $x[\tau]_{12\cdots h}^{12\cdots h}$, $\tau \in \mathscr{T}_+$ [resp. $\mathscr{T}_-$], depends only on $(|\tau_1|, ..., |\tau_k|)$, by continuity.

So there exists a function $\theta_+(r)$ [resp. $\theta_-(r)$], $r=(r_1,\dots,r_k)$, $m_i \geqslant r_i+\cdots+r_k$, $i=1,\dots,k$, with values $+1$ or -1 such that the strict inequality (5.5) holds if $\tau\in\mathcal{T}_+$ [resp. $\mathcal{T}_-$] is a partition of $h<n$ points in $I\setminus\{t_0\}$ or of $h=n$ points in I. In particular $\theta_+(r)\,W(X:r)(t)>0$ [resp. $\theta_-(r)\,W(X:r)(t)>0$] holds if $t\in I\setminus\{t_0\}$ and $W(X:r)$ is a primary m-Wronskian. By continuity from (5.5) the weak version of this inequality is satisfied by every m-Wronskian. Thus X is a $\mathcal{T}_+$ [resp. $\mathcal{T}_-$]-Markov system on $I\setminus\{t_0\}$ if θ_+ [resp. θ_-] is a signum function. We must therefore show that this function satisfies the relation (4.7).

We will show that, if $r_i=p_i+q_i$, $p_i>0$, $q_i\geqslant 0$, and $\varphi_i(q)=\theta_\pm(qe^i)$, then

$$\begin{aligned}\theta_\pm&(0,\dots,0,r_i,r_{i+1},\dots,r_k)\\&=\theta_\pm(0,\dots,0,q_i,r_{i+1},\dots,r_k)\,\varphi_i(r_i+\cdots+r_k)\\&\quad\times\varphi_i(q_i+r_{i+1}+\cdots+r_k)(\pm 1)^{p_i(r_{i+1}+\cdots+r_k)}\end{aligned}\tag{5.11}$$

with the alternative $+$ or $-$ being chosen consistently throughout. The cases $p_i=r_i$, $q_i=0$ of (5.11), $i=1,\dots,k-1$, give (4.7).

It suffices to prove (5.11) when $p_i=1$, $q_i=r_i-1$ since successive applications of that result give the formula for $1\leqslant p_i\leqslant r_i$. Also the proof is only given when $i=1$; the proof for $i>1$ is identical except for the zeros preceding r_i in the argument of $\theta_\pm$.

Consider a partition $\tau\in\mathcal{T}_\pm$ of $h\leqslant n$ points in $I\setminus\{t_0\}$ with $|\tau(t)|=|\tau_1(t)|=1$. Then, if t is close to t_0,

$$x[\tau]^{12\cdots h}_{12\cdots h}=x[\beta_t\tau]^{12\cdots h-1}_{12\cdots h-1}\,(-1)^{h-1}\,(\pm 1)^{|\tau_1|-1}\,x_1^h(t)[1+o(1)],\tag{5.12}$$

where $\beta_t\tau$ is the partition of $h-1$ points obtained by omitting t from τ. This formula is verified by expanding $x[\tau]^{12\cdots h}_{12\cdots h}$ by the row $[x_1^1(t),\dots,x_1^h(t)]$, which is row 1 when we are considering θ_+ and t is the closest point to t_0 and is row $|\tau_1|$ in the case of θ_-. The verification also uses the fact that $x_1^j(t)=o(x_1^h(t))$, $t\to t_0$, when $j<h\leqslant m_1$. Note that this is where condition (4.9) is needed; when proving (5.11) for $i>1$, one needs $x_i^j(t)=o(x_i^h(t))$, $t\to t_0$, when $j<h\leqslant m_i$. From (5.12), choosing $\tau\in\mathcal{T}_+$ so that $|\tau_i|=r_i$, $i=1,\dots,k$, and from the fact that the strict inequality (5.5) is satisfied, we infer

$$\theta_\pm(r_1,\dots,r_k)=\theta_\pm(r_1-1,r_2,\dots,r_k)(-1)^{h-1}\,(\pm 1)^{r_1-1}\operatorname{sgn}x_1^h(t),\qquad t\to t_0,\tag{5.13}$$

if $r_1>0$, $h=r_1+\cdots+r_k$. Choosing τ so that $|\tau_1|=|\tau|=h$ in (5.11), we find

$$\varphi_1(h)=\varphi_1(h-1)(-1)^{h-1}\,(\pm 1)^{h-1}\operatorname{sgn}x_1^h(t),\qquad t\to t_0,\tag{5.14}$$

since $\varphi_1(h) = \theta_{\pm}(h, 0, ..., 0)$. Therefore, from (5.13), (5.14),

$$\theta_{\pm}(r_1, ..., r_k) = \theta_{\pm}(r_1 - 1, r_2, ..., r_k)\, \varphi_1(h)\, \varphi_1(h-1)(\pm 1)^{h-r_1}$$

which is the case $i = 1$, $p_1 = 1$, $q_1 = r_1 - 1$, of (5.11).

6. Conclusion

The results of this paper could be formulated in a more general setting than that which we have used here. For example, we might consider spaces $\mathscr{X}$ of vector-valued functions in which zeros of various components are considered in different ways. If a certain component x_j of x is a step function on J, we might define zeros in terms of differences as in Hartman [7]. We might also consider zeros of some components in terms of families of quasidifferential operators in the sense of Zettl [22]. These changes would necessitate a more general idea of the Wronskian determinant with the derivatives of the appropriate components being replaced by differences and quasidifferential operators. The essential character of the development based on the formulas (5.1), (5.2) would not be much different.

Another aspect of Pólya's paper [20] is that it establishes the equivalence of disconjugacy of L in (1.1) and the existence of a factorization of L into a product of n first order operators. It would be interesting to find a similar criterion for $\mathscr{T}_{\pm}$-disconjugacy. However, we have not found a further restriction on the Pólya factorization of L which would give a necessary and sufficient condition even for the disfocality of L.

References

1. W. A. Coppel, "Disconjugacy," Lecture Notes in Mathematics, Vol. 220, Springer-Verlag, New York/Berlin, 1971.
2. U. Elias, Minors of the Wronskian of the differential equation $L_n y + p(x)y = 0$, *Proc. Roy. Soc. Edinburgh Sect. A* **106** (1987), 341–359.
3. S. Friedland, "Nonoscillation, Disconjugacy and Integral Inequalities," Memoirs Amer. Math. Soc., Vol. 176, Providence, RI, 1976.
4. F. R. Gantmacher, "The Theory of Matrices," Vol. I, Chelsea, New York, 1960.
5. H. Gingold, Uniqueness of linear boundary value problems for differential systems, *Pacific J. Math.* **75** (1978), 107–136.
6. P. Hartman, "Ordinary Differential Equations," Wiley, New York, 1964.
7. P. Hartman, Difference equations, disconjugacy, principal solutions, Green's functions, complete monotonicity, *Trans. Amer. Math. Soc.* **246** (1978), 1–30.
8. J. Henderson and L. Jackson, Existence and uniqueness of solutions of k-point boundary value problems for ordinary differential equations, *J. Differential Equations* **48** (1983), 373–385.

9. G. D. Jones, Growth properties of solutions of a linear differential equation, *Proc. Roy. Soc. Edinburgh Sect. A* **104** (1986), 127–136.
10. S. Karlin and W. J. Studden, "Tchebycheff Systems with Applications in Analysis and Statistics," Interscience, New York, 1976.
11. W. J. Kim, Disconjugacy and disfocality of differential systems, *J. Math. Anal. Appl.* **26** (1969), 9–19.
12. W. J. Kim, Properties of disconjugate linear differential operators, *J. Differential Equations* **43** (1982), 369–398.
13. D. London and B. Schwarz, Disconjugacy of complex differential systems and equations, *Trans. Amer. Math. Soc.* **135** (1969), 487–505.
14. J. S. Muldowney, Comparison theorems for linear boundary value problems, *SIAM J. Math. Anal.* **9** (1978), 943–955.
15. J. S. Muldowney, A necessary and sufficient condition for disfocality, *Proc. Amer. Math. Soc.* **74** (1979), 49–55.
16. J. S. Muldowney, On invertibility of linear ordinary differential boundary value problems, *SIAM J. Math. Anal.* **12** (1981), 368–384.
17. Z. Nehari, Disconjugate linear differential operators, *Trans. Amer. Math. Soc.* **129** (1967), 500–516.
18. Z. Nehari, Oscillation theorems for systems of linear equations, *Trans. Amer. Math. Soc.* **139** (1969), 339–347.
19. A. C. Peterson, Boundary value problems for an nth order linear difference equation, *SIAM J. Math. Anal.* **15** (1984), 124–132.
20. G. Pólya, On the mean-value theorem corresponding to a given linear homogeneous differential equation, *Trans. Amer. Math. Soc.* **29** (1922), 312–324.
21. B. Schwarz, Disconjugacy of complex differential systems, *Trans. Amer. Math. Soc.* **125** (1966), 482–496.
22. A. Zettl, Formally self-adjoint quasi-differential operators, *Rocky Mountain J. Math.* **5** (1975), 453–474.